Basics of Set Theory

Set Theory (Math and Music)

(note: figures and bibliography follow text)

Set theory belongs to a branch of mathematics known as *discrete* mathematics. Discrete mathematics focuses on fixed, discontinuous numbers in contrast to, for example, algebra and calculus that cover the continuous domain of all real numbers. The study of prime numbers represents a good example of discrete mathematics in that only numbers that are divisible by themselves and 1 qualify as prime. Discrete mathematics has many branches—number theory, game theory, group theory, and so on—as well as set theory.

Mathematical set theory—along with logic and predicate calculus—represents one of the axiomatic foundations of mathematics. A mathematical set is denoted by numbers contained in braces (curly brackets) and separated by commas as in {0,1,2}. Set theory relates sets in many ways. For example, the set {0,1,2} is a subset of the set {0,1,2,3,4}, in that all three members of the first set belong to the second set (common membership). The set {5,6,7} is not a subset of the set {0,1,2,3,4}, since none of the members of the first set belong to the second set. Such simple comparisons represent one of the manifold ways in which set theory relates sets according to membership.

Figure 3.1 presents several methods in which sets can be more formally compared. The notation $12 \in \{8,10,12\}$, for example, indicates that 12 is a member of the set {8,10,12}. Conversely, the notation $11 \notin \{8,10,12\}$ states that 11 does not belong to the set {8,10,12}. The notations $\subset$ and $\subseteq$ indicate two forms of subsets—sets whose members all belong to another set. A proper subset ($\subset$) refers to a range of possible subsets belonging to another set that do not include exactly that set itself. In contrast, the second notation for a subset here ($\subseteq$ and *improper* by inference) means that the range of possible subsets of one set *does* exactly include another set. The symbol $\not\subset$, in contrast, indicates that a set is not a subset of another set. The special symbol $\varnothing$ defines the empty set $\{\}$. Two of the many operative mathematical set theory notations, presented last in figure 3.1, represent important relationships between two sets: $\cup$ indicates a *union* of two sets creating a third set that contains all of the elements of both original sets, while $\cap$ indicates an *intersection* between two sets creating a third set containing only the elements that both the original sets have in common. The following five examples present logical applications of these symbols:

$\{1,4,5\} \subset \{0,1,2,3,4,5\}$

$\{1,4,5\} \subseteq \{1,4,5\}$

$\{1,4,5\} \not\subset \{0,2,5\}$

$\{1,4,5\} \cup \{0,4,5\} = \{0,1,4,5\}$

$\{1,4,5\} \cap \{0,4,5\} = \{4,5\}$

Combinations of these and other relationships in mathematical set theory produce extremely valuable results and principles that impact all of the various forms of mathematics. The following example provides a simple demonstration, with sets here given variable letter names so that the examples extend beyond particular sets to sets in general:

if

$A \subseteq B \cap C$

then

$A \subseteq B$

and

$A \subseteq C$.

In order to more easily understand such set theory operations, John Venn (1834-1923) invented a visual process (1880) that demonstrates set containments and overlaps. Figure 3.2 presents a Venn diagram of the just-presented logical deductions. Since A is a subset of the intersection of B and C ($A \subseteq B \cap C$), then A is also a subset of B ($A \subseteq B$) and a subset of C ($A \subseteq C$).

This brief introduction to mathematical set theory does not do justice to this extraordinary field of study. I have limited this discussion here simply because musical set theory does not typically use these symbols or operations. However, musical set theory does invoke the principles of mathematical set theory as well as including many of its own unique comparison techniques, as we shall soon see. For those musicians interested in pursuing the mathematical side of set theory in more detail, several dozens of good books await your study, several of which I include in the bibliography to this book (Devlin 1993; Ferreirós 1999; Halmos 1974; Lawvere and Rosebrugh 2002; Levy, 1979).

Musical set theory was initially introduced by Milton Babbitt (1960 and 1961), embellished by Allen Forte (1973), and further developed by John Rahn (1980) and George Perle (1991). Babbitt was

> ". . . concerned primarily with those set properties—pitch class and intervallic, order-preserving and merely combinational—and those relationships between and among forms of the set which are preserved

under the operations of the system, and which—in general—are independent of the singular structure of a specific set. Here, to the end of discovering certain compositional consequences of set structure, the concern will be with those attributes of set structure which maintain under the systematic operations only by virtue of the particular nature of a set, or of the class of sets of which it is an instance, together with a particular choice of operations." (Babbitt 1961, p. 129)

Musical set theory follows many of the same tenets of mathematical set theory. However, several special conditions apply in musical set theory that do not apply in mathematical set theory. For example, musical set theory invokes the notion of "pitch class" (presented briefly in chapter 2), where register (octave) no longer applies and where the pitch C-natural equals 0, C-sharp equals 1 . . . B-natural equals 11. The process of reducing out the register of a pitch (i.e., all C-naturals belonging to the pitch class 0) is often termed *modulo 12*, modulo being a mathematical process that returns only the remainder when dividing one number by another number. No pitch should appear twice in this notation. Thus, all doublings as well as registration disppears when reducing pitches to pitch-class sets. Representing pitch in this way initiates a process that attempts to reveal similarities between sets of pitches that appear quite different either in number form and/or in musical notation.

To help differentiate musical set theory from mathematical set theory, musical set theory typically uses brackets rather than braces for set notation as in [0,4,7], the set for a C-major triad in root position. Other easily recognizable sets include the C-minor triad [0,3,7], C-dominant-seventh chord [0,4,7,t] where "t" represents the number 10 ("e" represents the number 11) to maintain a single digit/letter symbol for each of the eleven pitch classes.

In mathematical set theory, the order of the elements within a set is irrelevant. However, the notion of ordered and unordered sets *is* important in musical set theory. *Ordered* sets follow the order of pitches and pitch classes found in the music under analysis. On the other hand, *unordered* sets follow ascending order, ignoring the order of pitches and pitch classes found in the music being analyzed. A simple example may be useful here. As noted in chapter 2, pitches are typically represented as 60 for middle C, 61 for middle C-sharp, and so on. Thus, an ordered pitch set could appear as [66,69,62], given that this represents the order of these pitches in the grouping selected from the music for this set. The ordered pitch-class version of the [66,69,62] set would then be [6,9,2]. The unordered pitch-class version of the [66,69,62] set would then be [2,6,9].

In traditional music theory, triads are typically analyzed from their root pitches upwards. Root positions of chords generally share a common feature—they have a smaller range between their outer pitches. Figure 3.3 provides an example of this. Here we see a D-major triad in its three incarnations: root position, first inversion, and second inversion. Note that the perfect fifth between the outer pitches of the root-position chord span a smaller distance than the minor sixth and major sixth of the two inversions. The same is true in differentiating what is called the "normal" form of pitch-class sets, with the exception that smaller intervals between inner pitch-classes can also count (described shortly), though the larger outer pitches of the set take precedence.

As example, the ordered set [6,9,2] has a distance of 8 between its outer pitch classes 6 and 2. This is figured by incrementally counting upward from pitch-class 6 to pitch-class 2, beginning again at 0 after 11 (or "e") as in 7, 8, 9, t, e, 0, 1, 2, or eight steps. This counting upward is important, as counting downward from 6 to 2 does not include pitch-class 9, necessary since we want to include all members of the set in our computations.

Reviewing figure 3.3 while reading the following description may help readers understand these machinations more clearly. Unordering the set [6,9,2] creates a total of three sets with furthest separated numbers inclusive of the remaining pitch class: the original [6,9,2], [9,2,6], and [2,6,9], counting upward from the lowest pitch class in each case. The pitch-class set [6,9,2] has a distance of eight between its outer pitch classes 6 and 2, as previously shown. The pitch-class set [9,2,6] has a distance of nine between its outer pitch classes 9 and 6, proved by counting upwards from pitch-class 9 to pitch-class 6. However, the pitch-class set [2,6,9] has a distance of only seven between its outer pitch classes 2 and 9. Thus, the pitch-class set [2,6,9] represents the normal form of the ordered set [6,9,2]. A glance back at figure 3.3 proves this yet further, as the root position D-major chord represents pitch classes 2, 6, and 9 in that order from the lowest pitch upward.

Using a traditional twelve-hour analog clock face provides a much easier way to visualize numerical pitch-class relationships (see figure 3.4) and to compute the normal form of sets. The internal "x"s here denote the pitch-class set [6,9,2], called "unordered" at this point since once placed on the clock face the original order of the pitches in the music no longer be ascertained. The set [2,6,9] clearly represents the normal form since beginning and counting from pitch-class 6 produces an 8 distance clockwise between outer pitches, and beginning on 9 creates a 9 distance clockwise between outer pitches, both of which are larger than the 7 clockwise distance between the outer pitches of [2,6,9].

Figure 3.5 presents another set for normal form pitch-class analysis. The [1,6,t] pitch-class set shown here does not produce the smallest outer range, since the outer pitch classes produce a distance of 9, larger than the smaller range of 7 produced by [6,t,1].

Using clock faces to discover the various forms of a set is analogous to using Venn diagrams in mathematical set theory. While mainly graphic devices, both Venn diagrams in mathematics and clock faces in music make human analysis far easier and often even inviting.

Up to this point, we have been reading pitch-class sets clockwise. Reading sets counterclockwise adds a powerful comparative tool for discovering more similarities between apparently diverse pitch-class sets, particularly post-tonal pitch-class sets. This process, called mirror inversion, means that all intervals appear inverted in musical notation. In figure 3.5, two sets have the same distance between outer pitches: [6,t,1] figured clockwise, and [1,t,6] figured counterclockwise. In situations such as this, the pitch-class set with the smallest internal intervals packed toward the pitch of origin wins, Thus, pitch-class set [1,t,6] (read counterclockwise from 1) succeeds since it has an internal interval of 3 (1 to t read counterclockwise equals 3) rather than 4 (6 to t read clockwise). At this point, we transpose this smallest form ([1,t,6]) to 0 by reading the intervals counterclockwise and beginning with "0" replacing "1," creating what we call its "prime form," the set [0,3,7].

All chords with equivalent interval content become equivalent when presented in prime form. Thus, the pitch-class set [2,6,9] when figured as above on a clock face and transposed becomes [0,3,7]. This procedure, while it might seem somewhat artificial, closely resembles the process we use when analyzing for musical function in tonal music, where a pitch set in one register reduces to a V_7 in C major (e.g., [G,B,D,F]), and a pitch set in another register reduces to a V_7 in D major (e.g., [A,C#,E,G]), both having the same function. Since major and minor keys do not typically exist in post-tonal music as they do in tonal music, [0,3,7] represents all major and minor triads and their inversions.

This apparent contradiction should not be particularly bothersome since the reductive process is intended to reveal similarities, not equivalencies.

Figure 3.6 presents a straightforward example of using set theory to discover similarities in post-tonal music. The four chords in this example appear very different both in pitch content and in register. To a discerning ear, however, they sound similar in many ways. Each of these sets reduces to the same prime form [0,1,3,6,8,9] following the just-described processes. Figure 3.7 shows each chord in figure 3.6 represented on a clock face with the process used to discover the prime form provided in the figure legend.

There are, then, typically three forms of pitch-class sets: *ordered*, *normal*, and *prime*. The *ordered* form indicates that the order of pitches in the set matters (thus, [0,1,3] and [1,3,0] represent different sets even though they contain the same pitch classes). The *normal* form (unordered as it no longer reflects the order in the music) accounts for the normal inversions of, for example, triads, and places sets so that they cover the smallest overall range. The *prime* form then accounts for mirror inversions of sets, finding the smallest outer range, packing pitch classes toward the pitch class of origin, and transposing the result to begin on zero. These definitions generally follow those of Babbitt (1961) and Forte (1973, see particularly pp. 4-5). In summary, then, the following presents one instance of these three forms of the same pitch-class set:

[6,9,2]	ordered form
[2,6,9]	normal form
[0,3,7]	prime form.

The above set, initially a major triad in first inversion (ordered form), appears as a root position normal form, and then as a minor triad in inverted and transposed-to-zero form.

Unfortunately, the fact that all major and minor triads ultimately reduce to a minor triad in prime form, means that an unambiguous major triad expressed in post-tonal music for particular reasons can be lost during analysis. It is not that the prime form minor triad is objectionable, but that the normal form major triad cannot be clearly discerned from the minor triad in the prime form. Many analysts have reverted to a two form version called a *representative* form: T_n, read as normal form transposed to zero, and T_nI, read as normal form transposed with inversion taken into account (see Rahn 1980, pp. 75-6). Rahn also uses the term *type*, and others the terms *pitch-structure* (Howe 1965) and *chords* (Regener 1974). Morris (1986, see particularly p. 79) offers these and other classification systems of pitch-class sets. The reason for these extra forms is to account for works in which the two forms of many sets such as the major triad—original and inversion—appear in important places in the music being analyzed, and thus should be reflected in the analysis of that music. As well, valuable characteristics of same-prime-form sets can be revealed when comparing only normal forms (e.g., numbers of common tones, and so on [see Straus 2000, pp. 71-2]).

One simple solution that many of my theory colleagues and I have adopted—that of simply transposing the currently accepted normal form to zero, is called the t-normal form (for *transposed* normal form). Thus, the ordered pitch-class set [6,9,2] would translate to the following forms:

[2,6,9]	unordered form
[2,6,9]	normal form
[0,4,7]	t-normal form
[0,3,7]	prime form.

Keeping track, then, of the manners in which unordered, normal, t-normal, and prime forms interact with one another in music becomes much easier and more obvious. The interplay of t-normal and prime forms expressed in this way becomes immediately apparent in works where composers juxtapose pitch-class sets of the same prime form but different t-normal forms in their music.

From this point on in this book, then, I will be using both this zero-based t-normal form and the more traditional normal form. This fact will become increasingly important in the following discussions in this chapter when mathematical creation of sets helps the understanding of how logical new sets—otherwise considered unrelated—result from pairings of previously-appearing sets. Extending "normal order" to t-normal order has no direct effect on computer applications of music analysis. However, as will be seen in this and later chapters, the manner in which the programs accompanying this book on CD-ROM use this "t-normal order" has substantial effect on the ability to clearly describe a program's operation and the principles that enable that program to work effectively.

Translating pitch sets into t-normal pitch class sets can be accomplished in Lisp quite easily, as the following code demonstrates:

```
(defun translate-to-t-normal-pitch-class-set (set)
  (translate-to-pcs (sort-and-clean set)))

(defun sort-and-clean (set)
  (my-sort #'< (remove-duplicates (modulo12 set))))

(defun modulo12 (set)
  (if (null set)()
      (cons (mod (first set) 12)
            (modulo12 (rest set)))))

(defun translate-to-pcs (mod-set &optional (first-set (first mod-set)))
  (if (null mod-set)()
      (cons (abs (- first-set (first mod-set)))
```

```
            (translate-to-pcs (rest mod-set) first-set))))
```

The function `translate-to-t-normal-pitch-class-set` here acts as a top-level operator of the two functions `sort-and-clean` and `translate-to-pcs`. The function `sort-and-clean` simply maps `modulo12` (a simple recursive function that uses the Lisp primitive `mod` to reduce all elements of its set argument to within the range 0-11) on its argument, removes all duplicates, and then sorts the result into ascending order. The function `my-sort` is a simple function to bypass the side effects of Lisp's somewhat unpredictable standard `sort` function. The function `translate-to-pcs` then transposes the set to begin on 0 by subtracting the first element of the set from each of its members. The use of Lisp's `&optional` in the first line of `translate-to-pcs` allows for optional arguments that users do not have to use when calling the function, and which enable, here, the variable `first-set` to continue to represent the first element of `mod-set` even though `mod-set` is slowly diminishing in size due to recursion. The arguments to `&optional` come in lists with the first element the name of a variable and the second element the default data contained in that variable. Running this code (`my-sort` is available in almost every Lisp file on the accompanying CD-ROM) produces the following:

```
? (translate-to-t-normal-pitch-class-set  '(60 64 67))
(0 4 7)
? (translate-to-t-normal-pitch-class-set  '(60 63 67))
(0 3 7)
? (translate-to-t-normal-pitch-class-set  '(60 63 67 70))
(0 3 7 10)
? (translate-to-t-normal-pitch-class-set  '(64 67 79 84))
(0 4 7)
```

Note that the last data converted here contains octave doubling and an inversion that nonetheless convert to the same result as the first test.

Set theory analysis provides many other tools for understanding post-tonal music. For example, interval vectors (counts of all the intervals present in a set) provide interesting ways for analysts to relate prime forms of sets that may otherwise seem unrelated. Vectors are represented by six-digit counts of the intervals of a minor second, major second, minor third, major third, perfect fourth, and augmented fourth, with the remaining intervals in the octave considered mirror inversions of these intervals. Thus, the set [0,1,3] has the vector 111000, since 0,1 is a minor second, 0,3 is a minor third, and 1,3 is a major second (all of the possible intervals contained in the set). As an example of vector relationship, consider the two sets [0,1,3,7] and [0,1,4,6], both of which have the same vector 111111. Other pitch-class set comparison techniques include similar but not equivalent prime forms of pitch-class sets indicating variation techniques in use. Analyzing interval sets as well as pitch-class sets can help analysts discover interesting contrasts and similarities in post-tonal music. Processes such as these represent but a few of the ways that musical set theory can aid in our fundamental understanding of post-tonal music.

While musical set theory does not reveal function in the way that tonal analysis does in tonal music, using musical set theory to decipher similarities between complex and otherwise unrevealing groupings of pitches often provides insights into music otherwise considered impenetrable. The manner in which composers limit their compositions to just a few of the 208 possible prime forms of chords between trichords (three-pitch groupings) and nonachords (nine-pitch groupings) indicates that set theory provides a valuable tool for analyzing post-tonal music. (The forms of chords—between trichords and nonachords (4-8)—have the names tetrachords, pentachords, hexachords, septachords, and octachords respectively.)

One of the biggest problems analysts face when using set theory to analyze post-tonal music is how to group music into appropriate collections for revealing the optimum number of set relationships. Computer programs can aid significantly in this process by using extraordinary accuracy and speed to remove the drudgery from what can otherwise be an enormously time-consuming process of trial and error. Even brute force computer programs that simply compare sets of ever decreasing sizes to find the sets that appear most often, can minimize the effort of what would otherwise take an analyst weeks or even months by hand. Of course, such computer programs cannot make intuitive leaps or musical decisions. They do, however, offer analysts a palette of possibilities from which they can then choose the most promising alternative.

Before describing more innovative programs for analyzing post-tonal music, I will first demonstrate a program (`define-and-lexicon-all-patterns` in the file called *database* in the folder called *sets database*) that groups and analyzes music for sets and—possibly more importantly—set variances, returning its best guess for one or more sets as the basis of the music under study. This program uses a straightforward artificial intelligence technique known as pattern matching and includes several processes that will become important to understand as this book progresses. The primary focus of these processes involves separating grouping and pitch-class matching programs into discrete tasks and only comparing groupings that match. This focus will eliminate the need to match *all* possible groupings, thus reducing the time necessary for analysis and increasing the potential for varying the parameters for subsequent analyses. Furthermore, we can collect similar—in whatever ways we wish to define—pitch-class patterns and include them as well by selecting other, related lexicons. Thus, by collecting patterns before comparing them and carefully distributing them into appropriately-named lexicons, we not only speed the process of pitch-class matching, we also make a wide range of pitch-class matching possible without having to redo the grouping process.

Pattern matching in this way reveals a wide variety of possibiloities not otherwise evident to one that requires both grouping and matching simultaneously.

As mentioned previously, equivalent but transposed pitch-class sets can be represented by T_n, where "T" represents the word transposition and "n" indicates the distance between two fundamentally equivalent pitch-class sets. For example, the normal-form sets [2,4,5,7] and [5,7,8,t] have three half-steps separating them, the second set therefore having a T_3 relationship to the first set. In this book, we simply transpose both sets to the same *t-normal* form. In either case, the process involves addition and/or subtraction of two assumed equivalent pitch-class sets. The notion of adding and subtracting the elements of two *non-equivalent* sets to achieve a third set, however, does not seem so natural. For example, consider the same pitch-class set above ([2,4,5,7]) added to the pitch-class set ([2,5,9,t]). The result ([4, 9, 14, 17]), when reduced modulo 12 and transformed to t-normal form, produces a new pitch-class set of [0,2,3,7], similar in many ways, but not equivalent to, the two sets that produced it. While this creation of new pitch-class sets from the addition or subtraction of two dissimilar pitch-class sets may seem far-fetched for analysis, when three different pitch-class sets appear as the primary sets of a work under analysis, and two of these add or subtract to produce the third, the process seems more purposeful.

Multiplying and dividing pitch-class sets by each other may seem even a more remote process than adding and subtracting these sets. However, these processes become immediately less remote upon encountering them in music. As example, figure 3.8 presents the pickup and opening four bars of Schoenberg's Op. 19, No. 6 from *Sechs Kleine Klavierstücke*. The t-normal forms of the first two trichords here, [0,3,5] and [0,5,7], show little relation to the third trichord t-normal form [0,1,8] ([0,1,4] in prime form). However, when the [0,3,5] and [0,5,7] t-normal forms are multiplied together (i.e., when multiplying the aligned pitch classes of each set modulo 12), the new set results in the t-normal form [0,3,e], or [0,1,4] in the prime form. Thus, the three sets share a common multiple. I have included a small program called SetMath on the CD-ROM accompanying this book that accepts either normal forms or unordered pitch-class sets and provides a series of results in the following form:

```
(run-sets '(0 3 5) '(0 5 7))
Added sets: (0 8)
Subtracted sets 2 from 1: (0 10)
Subtracted sets 1 from 2: (0)
Multiply sets: (0 3 11)
```

Note the t-normal forms of the two input sets use parentheses instead of brackets and lack commas, side effects of using Lisp. Note also that multiplying and otherwise mathematically combining sets can cause duplications, resulting in fewer numbers of pitch classes in the output.

Before proceeding further, I remind readers here of my earlier comments about composer intent. What composers do or not intend to include in their music, while of interest, should not deter analysts from revealing discovered relations, no matter how unlikely these relations may seem to the perceived concept of the work as we know it. I further remind readers that analysis also informs listening. Whether a process that exists in music can be heard or not, should not deter us from appreciating its presence.

Up to this point and continuing until chapter 7, sets in this book have primarily been comprised of pitch classes. Readers should be reminded, however, that sets may consist of representations of any parameter of music. For example, rhythmic information such as sets of durations in thousands of a second [1000,4000,2000,8000], channel settings [1,4,2,3], dynamics [55,60,75,45], and so on, can all produce useful relationships when permuted and compared. I have even found interesting results when creating formal sets as in [a,a,b,b,c,a,b] when comparing the forms of entire works to one another. Such formal collections remind us that sets may consist of many different kinds of symbols other than numerical information, as in this case, where the alphabetical letters represent phrases of a particular type of musical thematic or harmonic data.

Musical set theory involves far more than I have described here. In fact, the principles expressed thus far in this chapter barely cover the fundamentals of what can be a very complex and often personal approach to understanding post-tonal music. However, given that this book attempts to cover a wide variety of approaches, I leave it to individual readers to explore musical set theory further, using books of their own choosing or those that I have included in the bibliography of this book (particularly Forte 1973; Lewin 1987; and Straus 1990).

Figure 3.1. Symbols representing several ways in which sets can be more formally compared.

$\in$ is an element

$\notin$ is not an element

$\subset$ is a *proper* subset

$\subseteq$ is a subset

$\not\subset$ is not a subset

$\varnothing$ the empty set; a set with no

$\cup$ union

$\cap$ intersection

Figure 3.2. A Venn diagram of the just-presented logical deductions.

Figure 3.3. Root positions of chords have a smaller range between their outer notes.

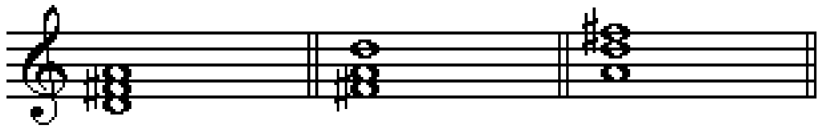


Figure 3.4. A clock face arranged to see pitch-class relationships more clearly.

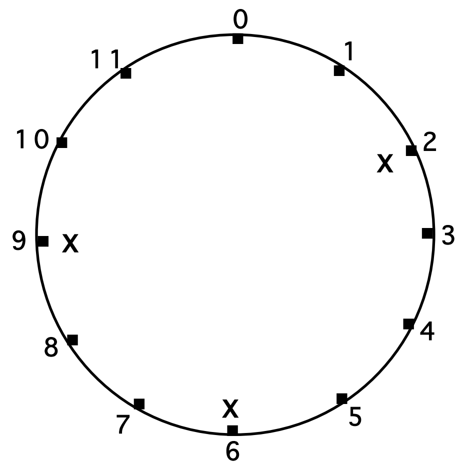


Figure 3.5. The [1,6,10] pitch-class set does not resolve to the pitch-class set [0,5,9] (9 distance when calculated from 0 or rotated to 0) because the smaller range of [6,10,1] (7 distance and equating to [0,4,7] when calculated from 0 or rotated to 0).

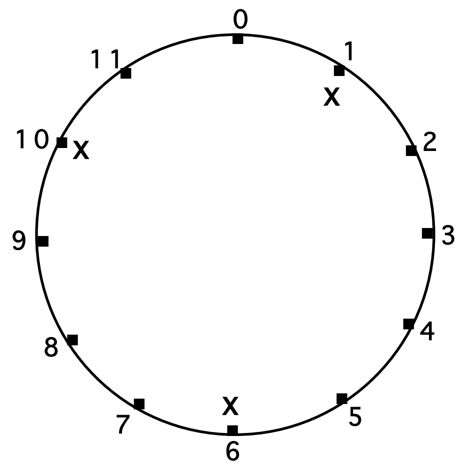


Figure 3.6. A straightforward example of the use of set theory to find similarities in post-tonal music.

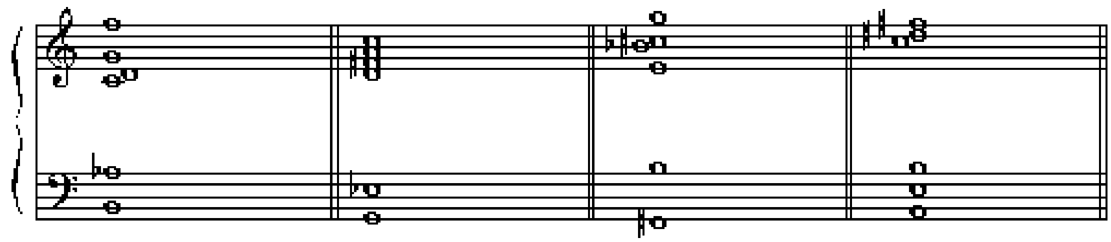


Figure 3.7. [11,0,2,5,7,8] is [0,1,3,6,8,9] when rotated one number clockwise; b) [6,7,9,0,2,3] is [0,1,3,6,8,9] when rotated six numbers clockwise; c) [7,6,4,1,11,10] is [0,1,3,6,8,9] when rotated seven numbers counterclockwise; d) [0.11.9.6.4.3] is [0,1,3,6,8,9] when rotated twelve numbers counterclockwise.

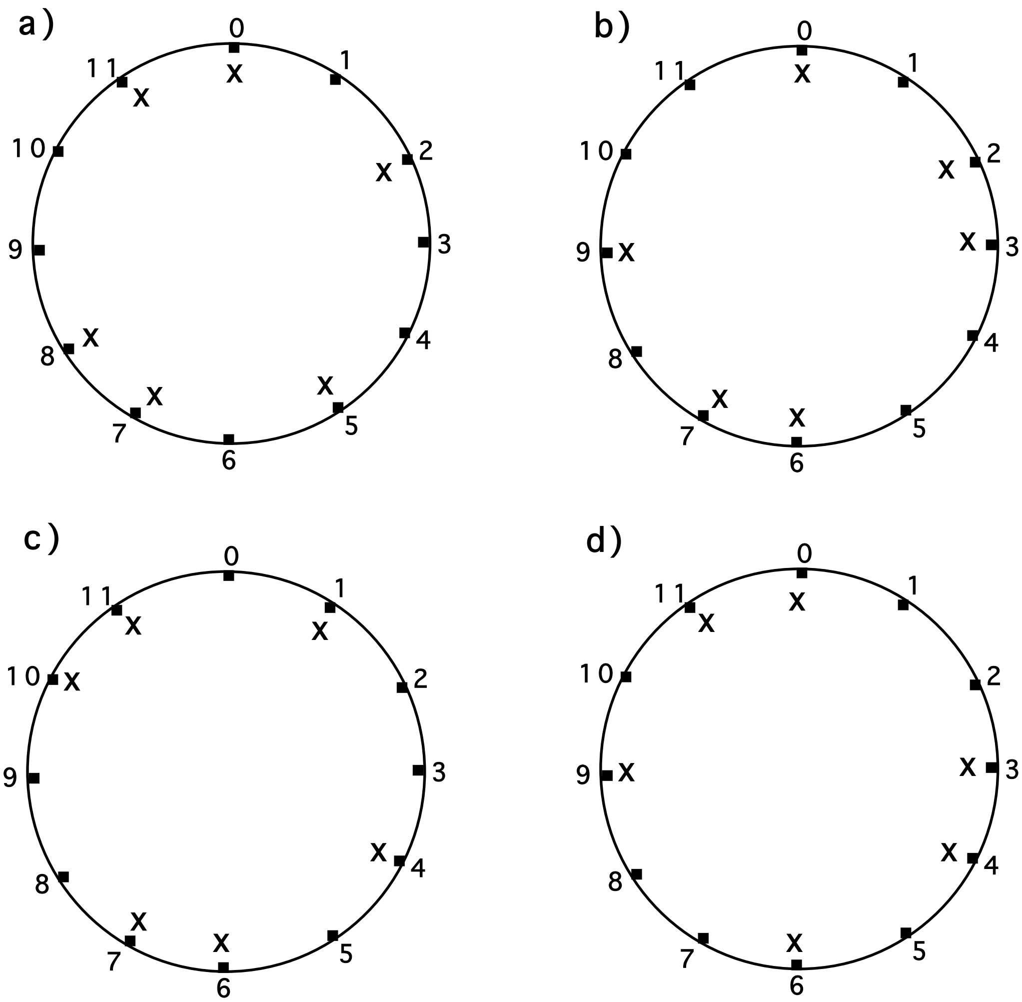
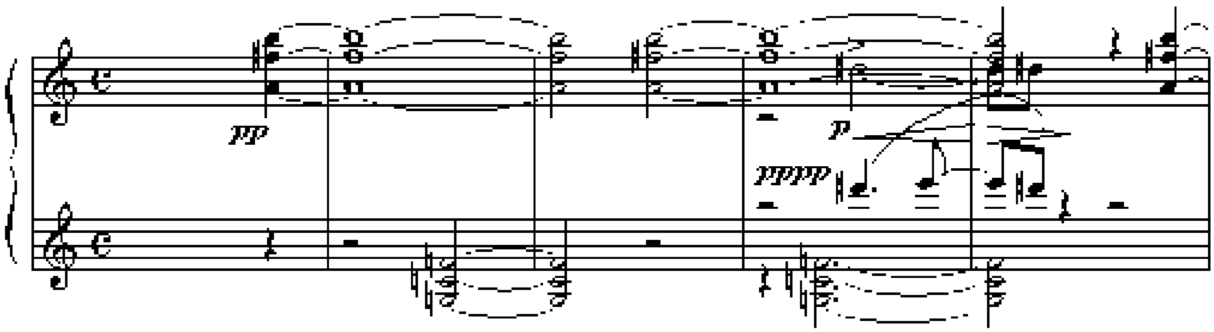


Figure 3.8. The pickup and opening four bars of Schoenberg's Op. 19, No. 6 (from *Sechs Kleine Klavierstücke*).



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